

# Independence of the metric in the fine $C^0$ -topology of a function space

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## 1 Introduction

Several topologies can be given to the space of continuous functions  $F(X, Y)$  from a topological space  $X$  into a metrizable space  $Y$ . One of them is defined as follows: let  $d$  be a bounded distance consistent with the topology of  $Y$ . A distance  $d^*$  can be defined on  $F(X, Y)$  by:

$$d^*(f, g) = \sup_{x \in X} d[f(x), g(x)].$$

The topology of  $F(X, Y)$  determined by  $d^*$  is called the  $d^*$ -topology. This topology depends not only on the topologies of  $X$  and  $Y$  but also on the metric  $d$  that is chosen for  $Y$ .

In §2 we define the fine  $C^0$ -topology on  $F(X, Y)$ . It is shown in [1, §5] that this topology does not depend on the chosen metric for  $Y$  in the case where  $X$  is paracompact. We show here (Theorem 1) the same result without imposing any condition on  $X$ .

In §3 some properties of the fine  $C^0$ -topology are mentioned and it is compared with other topologies.

## 2 The fine $C^0$ -topology

**Definition 1.** Let  $X$  be a topological space,  $(Y, d)$  metric,  $f: X \rightarrow Y$  continuous,  $\delta: X \rightarrow \mathbb{R}^+ = \{\text{positive reals}\}$  continuous.  $g: X \rightarrow Y$  continuous is a  $\delta$ -approximation of  $f$  if  $d[f(x), g(x)] < \delta(x)$  for all  $x \in X$ . The  $\delta$ -neighborhood of  $f$  is the set of all the  $\delta$ -approximations of  $f$ . If  $F(X, Y)$  is the set of continuous functions from  $X$  to  $Y$  and  $f \in F(X, Y)$ , we define a neighborhood of  $f$  as a subset of  $F(X, Y)$  that contains some  $\delta$ -neighborhood of  $f$ .

This defines a topology on  $F(X, Y)$  given that the intersection of the  $\delta$ -neighborhood of  $f$  and the  $\eta$ -neighborhood of  $f$  is the  $\min(\delta, \eta)$ -neighborhood of  $f$  and furthermore if  $g$  is in the  $\delta$ -neighborhood of  $f$  and  $\eta(x) = \delta(x) - d[f(x), g(x)]$ , then the  $\eta$ -neighborhood of  $g$  is contained in the  $\delta$ -neighborhood of  $f$ . We will call this topology the *fine  $C^0$ -topology* of  $F(X, Y)$ , and we denote by  $T_d(X, Y)$  the resulting topological space.

The next theorem establishes that the topology on  $T_d(X, Y)$  depends only on the topologies of  $X$  and  $Y$  and not on the metric on  $Y$ .

**Theorem 1.** *If  $d_1$  and  $d_2$  are two equivalent distances on  $Y$ , then  $T_{d_1}(X, Y)$  and  $T_{d_2}(X, Y)$  are the same topological space.*

*Proof.* Let  $W_1$  be a  $\delta_1$ -neighborhood of  $f: X \rightarrow Y$  on  $T_{d_1}(X, Y)$ . We will show that there exists  $W_2$ ,  $\delta_2$ -neighborhood of  $f$  in  $T_{d_2}(X, Y)$ , contained in  $W_1$ . The proof that for every  $\delta_2$  neighborhood of  $W_2$  of  $f$  there is a  $\delta_1$ -neighborhood contained in  $W_2$  being analogous.

We define  $\delta_2: X \rightarrow \mathbb{R}^+$  as

$$\delta_2 = G' \circ (f \times \delta_1),$$

where  $G'$  is a continuous function from  $Y \times \mathbb{R}^+$  to  $\mathbb{R}^+$  that satisfies:

$$G'(y, t) \leq \sup\{r | B_2(y, r) \subset B_1(y, t)\} = G(y, t)$$

( $B_i(y, r) = \{y' \in Y | d_i(y, y') < r\}$   $i = 1, 2$ ). Let's show that there is such a  $G'$ .

The function  $G$ , that is positive since  $d_1$  and  $d_2$  are equivalent, satisfies:

$$G(y', t) \geq G(y, t - \varepsilon) - \varepsilon \text{ if } d_i(y, y') < \varepsilon, \quad i = 1, 2.$$

Indeed, if  $w \in B_2(y', G(y, t - \varepsilon) - \varepsilon)$  and  $d_i(y, y') < \varepsilon$   $i = 1, 2$ ,

$$d_2(w, y') + \varepsilon < G(y, t - \varepsilon)$$

$$d_2(w, y) < G(y, t - \varepsilon)$$

$$B_2(y, s) \subset B_1(y, t - \varepsilon) \text{ with } d_2(w, y) < s < G(y, t - \varepsilon)$$

$$d_1(w, y) < t - \varepsilon$$

$$w \in B_1(y', t)$$

and

$$B_2(y', G(y, t - \varepsilon) - \varepsilon) \subset B_1(y', t)$$

that is

$$G(y', t) \geq G(y, t - \varepsilon) - \varepsilon.$$

Since  $G$  is not decreasing in the second variable, for  $(y, t) \in Y \times \mathbb{R}^+$  there is  $\varepsilon$  such that  $G(y, t - \varepsilon) - \varepsilon > 0$  and thus every point  $(y, t)$  of  $Y \times \mathbb{R}^+$  has a neighborhood in which  $G$  has a positive lower bound, namely the neighborhood  $[B_1(y, \varepsilon) \cap B_2(y, \varepsilon)] \times (a, \infty)$  with  $0 < a < t$  and  $G(y, a - \varepsilon) - \varepsilon > 0$ .

Hence, since the domain of  $G$  is paracompact, (every metrizable is paracompact), there is a locally finite open cover  $\{V_\alpha\}$  of this domain such that  $G$  has a positive lower bound  $\varepsilon_\alpha$  on each  $V_\alpha$ . Let  $\{\phi_\alpha\}$  be a partition of unity subordinated to  $\{V_\alpha\}$ . Define  $G'(y, t) = \sum_\alpha \varepsilon_\alpha \phi_\alpha(y, t)$ , a continuous function with values in  $\mathbb{R}^+$ .

We have that

$$G'(y, t) = \sum_{(y, t) \in V_\alpha} \varepsilon_\alpha \phi_\alpha(y, t) \leq \max_{(y, t) \in V_\alpha} \{\varepsilon_\alpha\} \leq G(y, t).$$

If  $W_2$  is the  $\delta_2$ -neighborhood of  $f$  in  $T_{d_2}(X, Y)$ ,  $W_2 \subset W_1$  since, if  $g \in W_2$ ,

$$d_2[g(x), f(x)] < \sup\{r | B_2(f(x), r) \subset B_1(f(x), \delta_1(x))\}$$

$$B_2(f(x), s) \subset B_1(f(x), \delta_1(x)) \text{ with}$$

$$d_2[g(x), f(x)] < s < \sup\{r | B_2(f(x), r) \subset B_1(f(x), \delta_1(x))\}$$

$$d_1[f(x), g(x)] < \delta_1(x)$$

and

$$g \in W_1$$

This completes the proof.

We can thus omit the subindex  $d$  in  $T_d(X, Y)$ .

### 3 Properties and comparison with other topologies

From now on  $Y$  will always denote a metrizable space.

The space  $T(X, Y)$  is always Tychonoff ( $T_1$  and completely regular) since if  $f \in T(X, Y)$  and  $\delta$  is any positive continuous function, the function from  $T(X, Y)$  to the reals defined by

$$g \mapsto \min \left[ \sup_{x \in X} \left\{ \frac{d[f(x), g(x)]}{\delta(x)} \right\}, 1 \right],$$

where  $d$  is a distance in  $Y$  consistent with the topology of  $Y$ , is continuous, has value 1 outside of the  $\delta$ -neighborhood of  $f$ , and has value 0 if and only if  $g = f$ .

A topology in  $F(X, Y)$  is called *admissible* if the function  $(f, x) \mapsto f(x)$  defined on  $F(X, Y) \times X$  turns out to be continuous when  $F(X, Y)$  is endowed with that topology. The fine  $C^0$ -topology and the  $d^*$  topology are admissible. The compact-open topology is coarser than the  $d^*$ -topology, which in turn is coarser than the fine  $C^0$ -topology [2].

In case  $X$  is compact these topologies are identical (see for example [3, Chap. 7, Theorem 11]). We will see that if  $X$  is  $T_4$ , not countably compact and  $Y$  has a subspace homeomorphic to the reals, then the three topologies are all different.

**Theorem 2.** *If  $X$  is countably compact (every open countable covering has a finite subcovering), the fine  $C^0$ -topology of  $F(X, Y)$  coincides with the  $d^*$ -topology. If  $X$  is  $T_1$  and normal,  $Y$  has a subspace homeomorphic to  $\mathbb{R}$  and  $T(X, Y)$  satisfies the first countability axiom, then  $X$  is countably compact.*

*Proof.* To show the first statement it suffices to show that every positive continuous function  $\delta$  on a countably compact space  $X$  has a positive lower bound.

If  $X$  is countably compact,  $\delta(X)$  is as well. A countably compact in  $\mathbb{R}$  is compact, so  $\delta(X)$  has a positive lower bound.

Assume now that  $X$  is  $T_1$ , normal and not countably compact, and that  $Y$  contains  $\mathbb{R}$  as a subspace. We will show that  $T(X, Y)$  does not satisfy the first countability axiom.

$T(X, \mathbb{R})$  is a subspace of  $T(X, Y)$  so it will suffice to show that  $T(X, \mathbb{R})$  does not satisfy the first countability axiom.

Since  $X$  is not countably compact there is a sequence of distinct points  $(x_i)$  without a cluster point in  $X$ .  $\{x_i\}$  is closed in  $X$  and discrete since  $X$  is a  $T_1$  space.

Let  $f \in T(X, \mathbb{R})$  be the function identically 0, and let  $\{\delta_i\}_{i=1,2,\dots}$  be any countable family of positive continuous functions on  $X$ . By the normality of  $X$  we can define  $\delta: X \rightarrow \mathbb{R}^+$  continuous such that  $\delta(x_i) = \frac{1}{2}\delta_i(x_i)$ ,  $i = 1, 2, \dots$

The  $\delta$ -neighborhood of  $f$  does not contain any  $\delta_i$ -neighborhood of  $f$ . Indeed, since  $X$  is normal, for every  $i$  there is  $g \in T(X, \mathbb{R})$  such that  $g(x_i) = \delta_i(x)$ ,  $g(X) = [0, \delta(x_i)]$  and  $g(X - V) = \{0\}$  where  $V$  is a neighborhood of  $x_i$  in which  $\delta_i$  is bigger than  $\frac{1}{2}\delta_i(x_i)$ .  $g$  is then in the  $\delta_i$ -neighborhood of  $f$  but not in the  $\delta$ -neighborhood of  $f$  (taking the usual distance in  $\mathbb{R}$ ).

Therefore the  $\delta_i$ -neighborhoods of  $f$  do not form a fundamental system of neighborhoods of  $f$ . This completes the proof of the theorem.

Condition  $T_1$  can not be omitted from the statement of the theorem since, if  $X$  is the set of natural numbers with the topology in which the open sets are the sets of the form  $\{1, 2, \dots, n\}$ , the empty set and the whole  $X$ , then  $X$  is normal, not countably compact and  $T(X, Y)$  satisfies the first countability axiom.

The proposition is also not valid if normal is replaced by completely regular. The following is a counterexample:  $X = \Omega' \times \omega' - \{(\Omega, \omega)\}$  where  $\Omega'$  and  $\omega'$  are the set of ordinals not bigger than the first uncountable ordinal  $\Omega$  and the set of ordinals not bigger than the first infinite ordinal  $\omega$  respectively, both spaces with the order topology.  $X$  is  $T_1$ , completely regular, not countably compact and  $T(X, Y)$  satisfies the first countability axiom.

**Corollary 1.** *If  $X$  is  $T_1$ , normal, not countably compact and  $Y$  has a subspace homeomorphic to  $\mathbb{R}$ , then the compact-open topology, the  $d^*$ -topology and the fine  $C^0$ -topology on  $F(X, Y)$  are all different. See [2, §3]*

## References

- [1] Whitehead, J.H.C.: Manifolds with transverse fields in euclidean space, Ann. of Math. vol. 73 (1961), pp 154-212.
- [2] Jackson, J.R.: Comparison of topologies on function spaces. Proc. Amer. Math. Soc. vol. 3 (1952), pp 156-158.
- [3] Kelley, J. L.: General topology (New York, 1955).